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Review Article

Bifurcation Analysis and Nonlinear Model Predictive Control of the Tri-Trophic Food Chain Model

Lakshmi N Sridhar*
Department of Chemical Engineering, University of Puerto Rico Mayaguez, Puerto Rico

***Corresponding author:** Lakshmi N Sridhar, Department of Chemical Engineering, University of Puerto Rico Mayaguez, PR 00681-9046, Puerto Rico.
Email: lakshmin.sridhar@upr.edu

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Abstract

Bifurcation analysis and Nonlinear model predictive control were performed on a tri- trophic food chain model. It is also demonstrated (both numerically and analytically) that the presence of the branch points was instrumental in obtaining the Utopia solution when the multiobjective nonlinear model prediction calculations were performed. Bifurcation analysis was performed using the MATLAB software MATCONT while the multi-objective nonlinear model predictive control was performed by using the optimization language PYOMO.

Keywords: Optimal control; Bifurcation; Tri-trophic food chain model

Background and Literature Review

Several workers have investigated the high nonlinearity of tri-trophic food chain models. Freedman and Wolkowicz [8] investigated predator-prey systems with group defense. Freedman and Ruan [7] showed the existence of. Hopf bifurcations in three-species food chain models. Chattopadhyay, Sarkar and, El Abdllaoui [3] developed a delay differential equation model on harmful algal blooms in the presence of toxic substances. Mukhopadhyay and Bhattacharyya [16] modeled phytoplankton allelopathy in a nutrient-plankton model with spatial heterogeneity. Bercia and Bercia [1] performed a bifurcation analysis of a model of a three-level food chain in a mangrove ecosystem. Nath, Kumari, Kumar and, Das [17] studied strategies to stabilize chaos in a tri-trophic food chain model. Kumar and Kumari [11] developed methods to control chaos in a three-species food chain model with fear effect. Upadhyay and, Naji [20] studied the dynamics of a three-species food chain model with Crowley–Martin type functional response. Upadhyay and, Raw [21]. Discussed the complex dynamics of a three- species food-chain model with Holling type IV functional response. Kumari [13] demonstrated a pattern formation in spatially extended tri-trophic food chain model systems: Kumar and Kumari [12] did a bifurcation study and pattern formation analysis of a tri-trophic food chain model with group defense and Ivlev-like nonmonotonic functional response.

Although several articles demonstrate the existence of bifurcations in the tri-trophic food chain models and thus highlight the nonlinearity, there has been no research so far studying the effect of the bifurcations on the optimization and control of these models.

In this work, bifurcation analysis and nonlinear model predictive control (MNLMPC) calculations are performed on the tri-trophic food chain model [12]. It is shown that the presence of the singularities in the tri-trophic food chain model cause the MNLMPC calculations to converge to the Utopia solution.

Objectives

The main objectives of this research are to

1. Perform bifurcation analysis on the tri-trophic food chain model

2. Perform MNLMPC calculations on the tri-trophic food chain model

3. Demonstrate that the presence of singularities in the tri-trophic food chain model results in the MNLMPC calculations converging to the Utopia solution.

The rest of this paper is organized as follows. The tri-trophic food chain model [12], the bifurcation analysis procedure, and the MNLMPC calculation strategy are described followed by an analysis demonstrating how the presence of the singularities cause the MNLMPC calculations to converge to the Utopia solution. The numerical results are then described followed by the conclusions.

Tri-Trophic Food Chain Model

In this model, r represents the birth rate of the prey, K is the carrying capacity of the prey, a is the half-saturation constant of the intermediate predator and top predator, and γ the maximum value that per capita reduction rate of the intermediate predator can attain. a_1 represents the conversion coefficient from prey to intermediate predator, γ_1 stands for the conversion coefficient from intermediate predator to top predator, d_1 the death rate of the intermediate predator in the absence of the prey while r represents the death rate of the top

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predator in the absence of the intermediate predator. β is the reciprocal of the density of the prey at which predation reaches its maximum

The parameter values are

$r = 0.7$; $k = 20$; $\alpha = 0.3$; $\alpha_1 = 0.15$; $\beta = 0.11$; $\gamma = 0.45$; $\gamma_1 = 0.99$; $d_1 = 0.23$; $d2 = 0.05$

a is both the control variable and the bifurcation parameter. $u_1(t)$, $u_2(t)$, $u_3(t)$ are the population densities of prey, intermediate predator and the top predator, The equations of this model are

$$\frac{au_1}{dt} = ru_1(au_1u_2e^{-\beta\mu 1} = f_1$$

$$\frac{au_2}{dt} = au_1u_2e^{-\beta u1} - \frac{\gamma u_2u_3}{a+u_2} - d_1u_2 = f_2 \quad (1)$$

$$\frac{du_3}{dt} = \frac{\gamma_1 u_2 u_3}{a+u_2} - d_2u_2 = f_3$$

Bifurcation Analysis

The existence of bifurcations in engineering problems that lead to multiple steady-state solutions is well known. Bifurcations that lead to multiple steady-state solutions are a) Branch Points and b) limit points. Both these bifurcation points are singularities where the Jacobian matrix of the set of steady-state equations is singular. However, at a branch point there are 2 distinct tangents at the singular point while at a limit point there is only one tangent at the singular point One of the most commonly used software to locate these bifurcations is CL_MATCONT [4,5] CL_MATCONT uses a continuation procedure implementing the Moore-Penrose matrix pseudo-inverse. A stationary solution of the model under is used to obtain, a set of points that corresponds to the equilibria of the ordinary differential equations. CL_MATCONT detects the singularities and bifurcation points in the solution path and obtains all the branches of the solutions starting from the bifurcation points.

CL_MATCONT detects singular points which are limit points, branch points and Hopf bifurcation points. Hopf bifurcation points do not cause multiple steady-states and therefore do not result in the formation of a maxima or minima. Limit and Branch points cause the existence of multiple solutions. Consider an ODE system

$$\dot{x} = f(x, \beta) \quad (2)$$

Where $x \varepsilon R^n$ Let the tangent plane at any point x be $[v1, v2, v3, v4, \ldots vn_{n+1}]$. Define matrix

A as

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial f_3} & \frac{\partial f_1}{\partial f_4} & \cdots\cdots\cdots & \frac{\partial f_1}{\partial f_n} & \frac{\partial f_1}{\partial \beta} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial f_3} & \frac{\partial f_2}{\partial f_4} & \cdots\cdots\cdots & \frac{\partial f_2}{\partial f_n} & \frac{\partial f_2}{\partial \beta} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial f_3} & \frac{\partial f_n}{\partial f_4} & \cdots\cdots\cdots & \frac{\partial f_n}{\partial f_n} & \frac{\partial f_n}{\partial \beta} \end{bmatrix} \quad (3)$$

The matrix A can be written in a compact form as

$$A = \left[B \middle| \frac{\partial f}{\partial \beta}\right] \quad (4)$$

The tangent surface must satisfy the equation

$$Av = 0 \qquad (5)$$

For both limit and branch points the matrix B must be singular. For a limit point (LP) the n+1[th] component of the tangent vector $v_{n+1} = 0$ and for a branch point (BP) the matrix $\begin{bmatrix} A \\ v^T \end{bmatrix}$ must be singular [9,14,15].

Multi-objective Nonlinear Model Predictive Algorithm

The Multiobjective Nonlinear Model Predictive Control Strategy (MNLMPC) method was first proposed by Flores Tlacuahuaz [6] and used by Sridhar [19]. This method does not involve the use of weighting functions, nor does it impose additional constraints on the problem unlike the weighted function or the epsilon correction method [16]. For a problem that is posed as

$$\min J(x, u) = (x_1, x_{2\ldots\ldots} x_k)$$
$$\text{subject to } \frac{dx}{dt} = F(x, u)$$
$$h(x,u) \le 0$$
$$x^L \le x \le x^U \quad (6)$$
$$u^L \le u \le u^U$$

The MNLMPC method first solves dynamic optimization problems independently minimizing/maximizing each any variable p_i individually. The minimization/maximization of p_i will lead to the values p_i^*. Then the optimization problem that will be solved is

$$\min\{p_i - p_i^*\}^2$$
$$\text{subject to } \frac{dx}{dt} = F(x, u)$$
$$h(x, u) \le 0 \quad (7)$$
$$x^L \le x \le x^U$$
$$u^L \le u \le u^u$$

This will provide the control values for various times. The first obtained control value is implemented and the remaining discarded. This procedure is repeated until the implemented and the first obtained control value are the same.

The optimization package in Python, Pyomo [10], where the differential equations are automatically converted to a Nonlinear Program (NLP) using the orthogonal collocation method [2]. The Lagrange-Radau quadrature with three collocation points is used and 10 finite elements are chosen to solve the optimal control problems. The resulting nonlinear optimization problem was solved using the solvers IPOPT [23], and confirmed as global solutions with Baron [20] To summarize the steps of the algorithm are as follows

1. Minimize/maximize p_i subject to the differential and algebraic equations that govern the process using Pyomo with IPOPT and Baron. This will lead to the value p_i^* at various time intervals t_i. The subscript i is the index for each time step.

2. Minimize $\{p_i - p_i^*\}^2$ subject to the differential and algebraic equations that govern the process using Pyomo with IPOPT and Baron. This will provide the control values for various times.

3. Implement the first obtained control values and discard the remaining.

Repeat steps 1 to 4 until there is an insignificant difference between the implemented and the first obtained value of the control variables or if the Utopia point is achieved.

The Utopia point is when $p_i - p_i^*$ for all i.

Effect of singularities (Limit Point (LP) and Branch Point (BP)) on MNLMPC

Let the minimization be of the variable p_1 lead to the value M_1 and the minimization of function p_2 lead to the value M_2. This is equivalent to minimizing $(p_1 - M_1)^2$ and $(p_2 - M_2)^2$. The subsequent multiobjective minimization will be of the function $(p_1 - M_1)^2 + (p_2 - M_2)^2$.

The multi-objective optimal control problem is

$$\min (p_1 - M_1)^2 + (p_2 - M_2)^2$$

$$subject\ to\ \frac{dx}{dt} = F(x, u)\ (1)$$

For all i,

$$\frac{d}{dx_i}((p_1 - M_1)^2 + (p_2 - M_2)^2) = 2(p_1 - M_1)\frac{d}{dx_i}(p_1 - M_1) + 2(p_2 - M_2)\frac{d}{dx_i}(p_2 - M_2)$$

(2)

At the Utopia point both $(p_1 - M_1)$ and $(p_2 - M_2)$ are zero. Hence

$$\frac{d}{dx_i}((p_1 - M_1)^2 + (p_2 - M_2)^2) = 0\ (3)$$

Now let us look at the co-state equation

$$\frac{d}{dt}(\lambda_i) = -\frac{d}{dx_i}((p_1 - M_1)^2 + (p_2 - M_2)^2) - g\lambda_i\ (4)$$

The first term in this equation is 0 and hence

$$\frac{d}{dt}(\lambda_i) = -g\lambda_i$$

$$\lambda_i(t_f) = 0\ (5)$$

If the set ODE $\frac{dx}{dt} = g(x, u)$ has a limit or a branch point, g_x is singular.

Hence there are two different vectors-values for $[\lambda_i]$ where $\frac{d}{dt}(\lambda_i) > 0$ and $\frac{d}{dt}(\lambda_i) < 0$.

In between there is a vector $[\lambda_i]$ where $\frac{d}{dt}(\lambda_i) > 0$. This coupled with the boundary condition $\lambda_i(t_f) = 0$ will lead to $[\lambda_i]$ which will make the problem an unconstrained optimization problem and the one and only solution for the unconstrained problem is the Utopia solution.

Numerical Results

The MATLAB software MATCONT was used to perform the bifurcation analysis which revealed the existence of a limit point, a branch point and a Hopf bifurcation point all on the same curve. For the co-ordinates $[u_1, u_2, u_3, a]$, the limit point (LP) occurred at a value of (10.909096 3.521336 40.727707 66.201119), a Branch Point (BP) occurred at a value of (1.887061 2.600669 -0.000000 48.892584) while a Hopf bifurcation point (H) occurred at a value of (9.246831 3.469177 41.455132 65.220534). The three points are shown in figure 1.

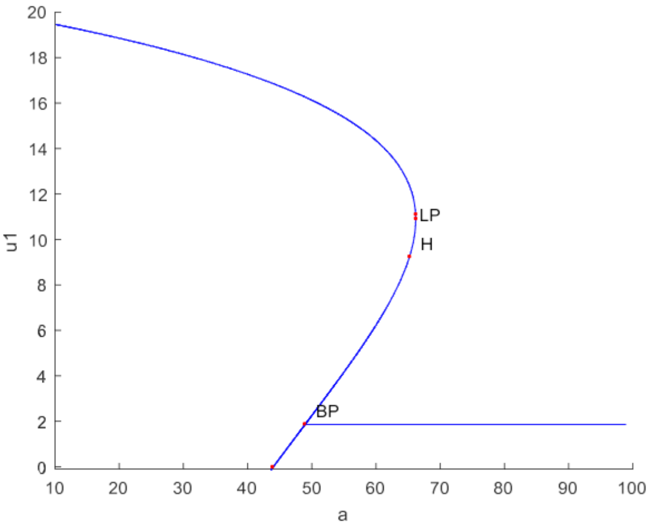

Figure 1:

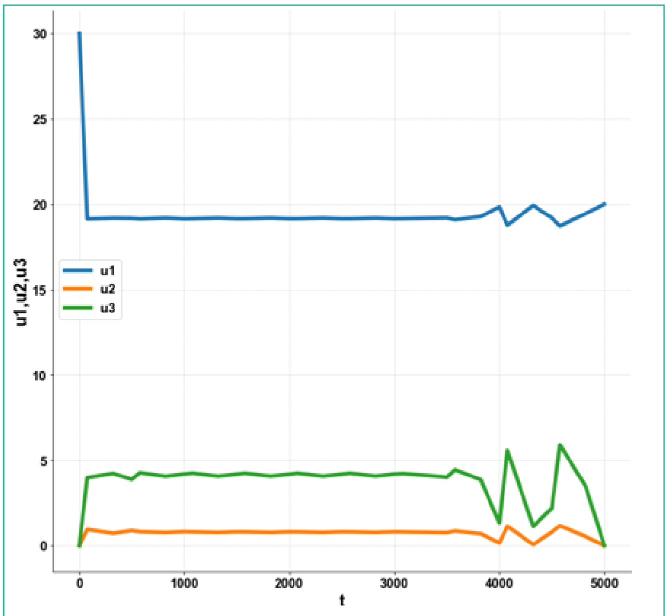

Figure 2:

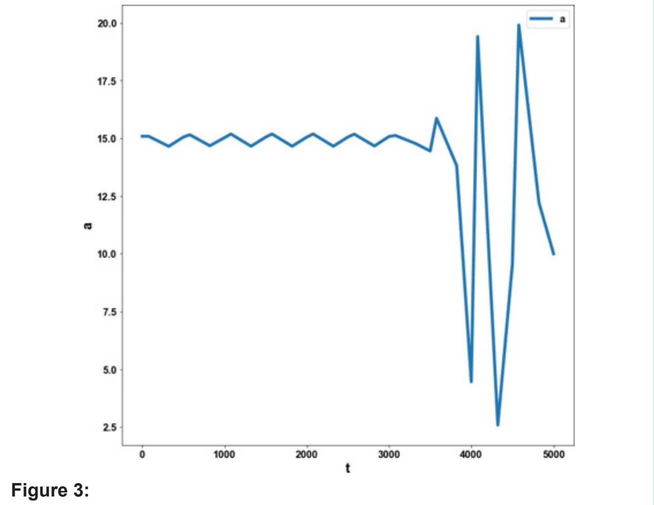

Figure 3:

For the MNLMPC calculations, u_1 was maximized and ($u_2 + u_3$) was minimized. The maximization of $\sum_0^{t_f} u_1$ resulted in a value of 50.0014, while the minimization of $\sum_0^{t_f} (u_2 + u_3)$ produced a value of 0. For the MNLMPC calculation the objective function $\sum_0^{t_f} ((u_2 + u_3) - 0)^2$ was minimized subject to the differential equations representing the Tri-Trophic Food Chain Model. This resulted in a value of 0 which is the Utopia solution and the obtained MNLMC value of the control variable a was 15.086.

Fig. 2 shows the $u_1(t)$, $u_2(t)$, $u_3(t)$ profiles while Figure 3 shows the control variable (a) profile.

Conclusions

This work re-emphasizes the highly nonlinear nature of the tri-trophic food chain model demonstrating the existence of the limit point, branch point, and the Hopf bifurcation point on the same curve and for the same bifurcation parameter. However, the existence of these bifurcation points is not a cause for concern as these singular points actually aid in the multiobjective nonlinear model predictive control calculations to converge to the best solution (Utopia point).

Author Statements

Conflict of Interest

The author, Dr. Lakshmi N Sridhar has no conflict of interest. The ethical statement is not applicable.

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